

ALMOST INVARIANT SETS

M.J. DUNWOODY

ABSTRACT. A short proof of a conjecture of Kropholler is given. This gives a relative version of Stallings' Theorem on the structure of groups with more than one end. A generalisation of the Almost Stability Theorem is also obtained, that gives information about the structure of the Sageev cubing.

1. INTRODUCTION

Let G be a group. A subset A of G is said to be *almost invariant* if the symmetric difference $A + Ag$ is finite for every $g \in G$. In addition A is said to be *proper* if both A and $A^* = G - A$ are infinite. The group G is said to have more than one end if it has a proper almost invariant subset.

Theorem 1.1. *A group G contains a proper almost invariant subset (i.e. it has more than one end) if and only if it has a non-trivial action on a tree with finite edge stabilizers.*

This result was proved by Stallings [13] for finitely generated groups and was generalized to all groups by Dicks and Dunwoody [3]. The action of a group G on a tree is *trivial* if there is a vertex that is fixed by all of G . Every group has a trivial action on a tree.

Let T be a tree with directed edge set ET . If e is a directed edge, then let \bar{e} denote e with the reverse orientation. If e, f are distinct directed edges then write $e > f$ if the smallest subtree of T containing e and f is as below.



Suppose the group G acts on T . We say that g *shifts* e if either $e > ge$ or $ge > e$. If for some $e \in ET$ and some $g \in G$, g shifts e , then G acts non-trivially on a tree T_e obtained by contracting all edges of T not in the orbit of e or \bar{e} . In this action there is just one orbit of edge pairs. Bass-Serre theory tells us that either $G = G_u *_{G_e} G_v$ where u, v are the vertices of e and they are in different orbits in the contracted tree T_e , or G is the HNN-group $G = G_u *_{G_e}$ if u, v are in the same G -orbit. If either case occurs we say that G splits over G_e .

If there is no edge e that is shifted by any $g \in G$, (and G acts without involutions, i.e. there is no $g \in G$ such that $ge = \bar{e}$) then G must fix a vertex or an end of T . If the action is non-trivial, it fixes an end of T , i.e. G is a union of an ascending sequence of vertex stabilizers, $G = \bigcup G_{v_n}$, where v_1, v_2, \dots is a sequence of adjacent vertices and $G_{v_1} \leq G_{v_2} \leq \dots$ and $G \neq G_{v_n}$ for any n .

Thus Theorem 1.1 could be restated as

Theorem 1.2 ([13], [3]). *A group G contains a proper almost invariant subset (i.e. it has more than one end) if and only if it splits over a finite subgroup or it is countably infinite and locally finite.*

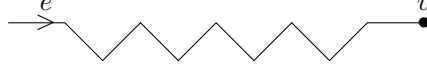
The if part of the theorem is fairly easy to prove. We now prove a stronger version of the if part, following [2].

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Let H be a subgroup of G . A subset A is H -finite if A is contained in finitely many right H -cosets, i.e. for some finite set F , $A \subseteq HF$. A subgroup K is H -finite if and only if $H \cap K$ has finite index in K . Let T be a G -tree and suppose there is an edge e and vertex v .

We say that e *points at* v if there is a subtree of T as below. We write $e \rightarrow v$.

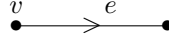


Let $G[e, v] = \{g \in G \mid e \rightarrow gv\}$.

If $h \in G$, then $G[e, v]h = G[e, h^{-1}v]$, since if $e \rightarrow gv$, $e \rightarrow gh(h^{-1}v)$.

It follows from this that If $K = G_v$, then $G[e, v]K = G[e, v]$. Also if $H = G_e$, then $HG[e, v] = G[e, v]$.

If $v = \iota e$, then $G_e = H \leq K = G_v$ and if $A = G[e, \iota e]$, then $A = HAK$.



Consider the set $Ax, x \in G$. If $g \in A, gx \notin A$, then $e \rightarrow gv, \bar{e} \rightarrow gxv$. This means that e is on the directed path joining gxv and gv . This happens if and only if $g^{-1}e$ is on the path joining xv and v . There are only finitely many directed edges in the G -orbit of e in this path. Hence $g^{-1} \in FH$, where F is finite, and $H = G_e$, and $g \in HF^{-1}$. Thus $A - Ax^{-1} = HF^{-1}$, i.e. $A - Ax^{-1}$ is H -finite. It follows that both $Ax - A$ and $A - Ax$ are H -finite and so $A + Ax$ is H -finite for every $x \in G$, i.e. A is an H -almost invariant set.

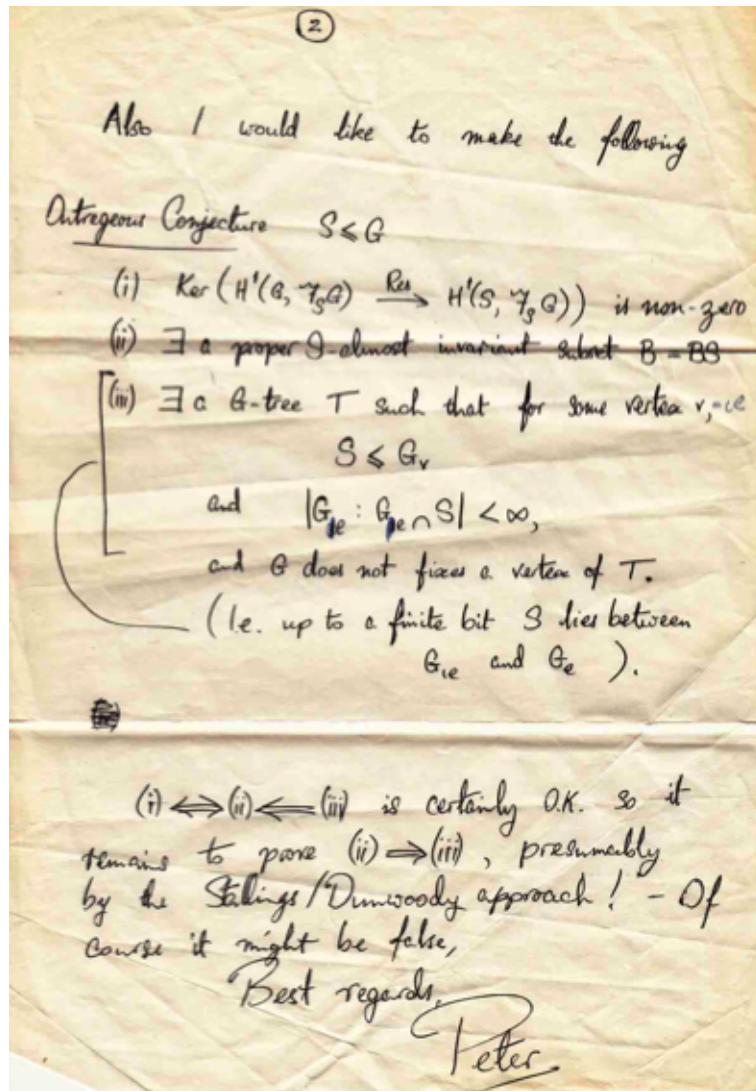
If the action on T is non-trivial, then neither A nor A^* is H -finite. We say that A is *proper*.

Peter Kropholler has conjectured that the following generalization of Theorem 1.1 is true for finitely generated groups.

Conjecture 1.3. *Let G be a group and let H be a subgroup. If there is a proper H -almost invariant subset A such that $A = AH$, then G has a non-trivial action on a tree in which H fixes a vertex v and every edge incident with v has an H -finite stabilizer.*

We have seen that the conjecture is true if H has one element. The conjecture has been proved for H and G satisfying extra conditions by Kropholler [8], Dunwoody and Roller [6], Niblo [10] and Kar and Niblo [7].

If G is the triangle group $G = \langle a, b \mid a^2 = b^3 = (ab)^7 = 1 \rangle$, then G has an infinite cyclic subgroup H for which there is a proper H -almost invariant set. Note that in this case G has no non-trivial action on a tree, so the condition $A = AH$ is necessary in Conjecture 1.3.



A discussion of the Kropholler Conjecture is given in [11]. I first learned of this conjecture in a letter Peter wrote to me in January 1988, a page of which is shown here.

We give a proof of the conjecture when G is finitely generated over H , i.e. it is generated by H together with a finite subset.

I am very grateful to Peter Kropholler for enjoyable discussions and a very helpful email correspondence.

2. INFINITE NETWORKS

Let X be an arbitrary connected simple graph. It is not even assumed that X is locally finite. Let $\mathcal{B}X$ be the set of all edge cuts in X . Thus if $A \subset VX$, then $A \in \mathcal{B}X$ if δA is finite. Here δA is the set of edges which have one vertex in A and one in A^* .

A ray R in X is an infinite sequence x_1, x_2, \dots of distinct vertices such that x_i, x_{i+1} are adjacent for every i . If A is an edge cut, and R is a ray, then there exists an integer N such that for $n > N$ either $x_n \in A$ or $x_n \in A^*$. We say that A separates rays $R = (x_n), R' = (x'_n)$ if for n large enough either $x_n \in A, x'_n \in A^*$ or $x_n \in A^*, x'_n \in A$. We define $R \sim R'$ if they are not separated by any edge cut. It is easy to show that \sim is an equivalence relation on the set ΦX of rays in X . The set $\Omega X = \Phi X / \sim$ is the set of edge ends of X . An edge cut A separates ends ω, ω' if it separates rays

representing ω, ω' . A cut A separates an end ω and a vertex $v \in VX$ if for any ray representing ω , R is eventually in A and $v \in A^*$ or vice versa.

We define a network N to be a simple, connected graph X and a map $c : EX \rightarrow \{1, 2, \dots\}$. If X is a network in which each edge has capacity 1, then $\mathcal{B}X$ is the set of edge cuts, and if $A \in \mathcal{B}X$, then $c(A) = |\delta A|$.

The following result is proved in [5].

Theorem 2.1. *Let $N(X)$ be a network in which X is an arbitrary connected graph. For each $n > 0$, there is a network $N(T_n)$ based on a tree T_n and a map $\nu : VX \cup \Omega X \rightarrow VT \cup \Omega T$, such that $\nu(VX) \subset VT$ and $\nu x = \nu y$ for any $x, y \in VX \cup \Omega X$ if and only if x, y are not separated by a cut A with $c(A) \leq n$.*

The network $N(T_n)$ is uniquely determined and is invariant under the automorphism group of $N(X)$.

Theorem 2.1 is proved by proving the following lemma. Let $\mathcal{B}_n X$ be the subring of $\mathcal{B}X$ generated by the cuts A such that $c(A) < n$

Lemma 2.2. *There is a uniquely defined nested set \mathcal{E}_n of generators of $\mathcal{B}_n X$, with the following properties:-*

- (i) *If G is the automorphism group of $N(X)$, then \mathcal{E}_n is invariant under G .*
- (ii) *For each $i < j$, $\mathcal{E}_i \subseteq \mathcal{E}_j$.*

We will only really be using Theorem 2.1 for networks in which every edge has capacity one.

Theorem 2.3. *Let X be a connected graph. There is a uniquely determined sequence of structure trees T_n and a map $\nu : VX \cup \Omega X \rightarrow VT \cup \Omega T$, such that $\nu(VX) \subset VT$ and $\nu x = \nu y$ for any $x, y \in VX \cup \Omega X$ if and only if x, y are not separated by a cut A with $|\delta A| \leq n$. Each tree T_n admits an action of the automorphism group of X .*

In this case $ET_n = \mathcal{E}_n$.

In any tree T if p is a vertex and Q is a set of unoriented edges, then there is a unique set of vertices P such that $v \in P$ then the geodesic $[v, p]$ contains an odd number of edges from Q . We then have $\delta P = Q$. Note that $\mathcal{B}T = \mathcal{B}_1 T$ and every element of $\mathcal{B}T$ is uniquely determined by the set Q together with the information for a fixed $p \in VT$ whether $p \in A$ or $p \in A^*$. The vertex p induces an orientation \mathcal{O}_p on the set of pairs $\{e, \bar{e}\}$ of oriented edges by requiring that $e \in \mathcal{O}$ if e points at p . For $A \in \mathcal{B}T$, A is uniquely determined by δA together with the orientation $\mathcal{O}_p \cap \delta A$ of the edges of δA .

In X it is the case that a cut A is uniquely determined by δA together with the information for a fixed $p \in VX$ whether $p \in A$ or $p \in A^*$.

Since $\mathcal{B}_n X$ is generated by $\mathcal{E}_n = ET_n$, the cut A can be expressed in terms of a finite set of oriented edges of T_n . This set is not usually uniquely determined. Thus if ν is not surjective, and v is not in the image of ν , and the set of edges incident with v is finite, then VX is the union of these elements in $\mathcal{B}X$. The empty set is the intersection of the complements of these sets. Orienting the edges incident with v towards v gives the empty set and orienting them away from v gives all of VX . However there is a canonical way of expressing an element of $\mathcal{B}_n X$ in terms of the generating set \mathcal{E}_n . To see this let $A \in \mathcal{B}_n X - \mathcal{B}_{n-1} X$. There are only finitely many $C \in \mathcal{E}_n$ with which C is not nested. This number is $\mu(A, \mathcal{E}_n) = \mu(A)$. We use induction on $\mu(A)$. Our induction hypothesis is that there is a canonically defined way of expressing A in terms of the \mathcal{E}_n . Any two ways of expressing A in terms of \mathcal{E}_n differ by an expression which gives the empty set in terms of \mathcal{E}_n . Such an expression will correspond to a finite set of vertices each of which has finite degree in T_n and none of which is in the image of ν . The canonical expression is obtained if there is a unique way of saying whether or not each such vertex is in the expression for A . Thus the canonical expression for A is determined by a set of vertices of VT which consists of the vertices of $\nu(A)$ together with a recipe for deciding for each vertex which is not in the image of ν whether it is in the expression for A .

Suppose $\mu(A) = 0$, so that A is nested with every $C \in \mathcal{E}_n$, and neither A nor A^* is empty. If $A \in \mathcal{E}_n$, then this gives an obvious way of expressing A in terms of the \mathcal{E}_n . If A is not in \mathcal{E}_n , then it

corresponds to a unique vertex $z \in VT_n$. Thus because $\mu(A) = 0$, A induces an orientation of the edges of \mathcal{E}_n . To see this, let $C \in \mathcal{E}_n$, then just one of $C \subset A, C^* \subset A, C \subset A^*, C^* \subset A^*$ holds. From each pair C, C^* we can choose C if $C \subset A$ or $C \subset A^*$ and we choose C^* if $C^* \subset A$ or $C^* \subset A^*$. Let \mathcal{O} be this subset of \mathcal{E} . Then If $C \in \mathcal{O}$ and $D \in \mathcal{E}$ and $D \subset C$, then $D \in \mathcal{O}$. This means that the orientation \mathcal{O} determines a vertex z in VT_n . Intuitively the edges of \mathcal{O} point at the vertex z . It can be seen that A or A^* will be the union of finitely many edges E of $\mathcal{E}_n = ET_n$, all of which have $\tau E = z$. If A is such a union, then we use this to express $A = C_1 \cup C_2, \dots \cup C_k$. If A is not such a union, but $A^* = C_1 \cup C_2, \dots \cup C_k$, then we write $A = (C_1 \cup C_2, \dots \cup C_k)^* = C_1^* \cap C_2^* \cap \dots \cap C_k^*$. Note that this gives a unique way of expressing cuts corresponding to a vertex z of finite degree not in the image of ν . The vertex z is included in the expression for A^* if and only if only finitely many cuts in \mathcal{E}_n incident with z and pointing at z are subsets of A . Suppose then that the hypothesis is true for elements $B \in \mathcal{B}_n X$ for which $\mu(B) < \mu(A)$. Let $C \in \mathcal{E}_n$ be not nested with A . Then $\mu(A \cap C) + \mu(A \cap C^*) \leq \mu(A)$. Thus each of $A \cap C$ and $A \cap C^*$ can be expressed in a unique way in terms of the \mathcal{E}_n . If at most one of these expressions involves C then we take the expression for A to be the union of the two expressions for $A \cap C$ and $A \cap C^*$. If both of the expressions involve C , then we take the expression for A to be the union of the two expression with C deleted. The expression obtained for A is independent of the choice of C . In fact the decomposition will involve precisely those C for which C occurs in just one of the decompositions for $A \cap C$ and $A \cap C^*$. We therefore have a canonical decomposition for A . To further clarify this proof observe the following. The edges C which are not nested with A form the edge set of a finite subtree F of T_n . If $EF \neq \emptyset$ we can choose C so that it is a twig of F , i.e. so that one vertex z of F is only incident with a single edge C of F . By relabelling C as C^* if necessary we can assume that $\mu(A \cap C) = 0$. The vertex determined by $A \cap C$ as above is z , and we have spelled out the recipe for if this vertex is to be included in the expression for A . The induction hypothesis gives us a canonical expression for $A \cap C^*$, which together with the expression for $A \cap C$ gives the expression for A .

3. RELATIVE STRUCTURE TREES

We prove Conjecture 1.3 in the case when G is finitely generated over H , i.e. G is generated by $H \cup S$ where S is finite.

First, we explain the strategy of the proof. Suppose that we have a non-trivial G -tree T in which every edge orbit contains an edge which has an H -finite stabiliser, and suppose there is a vertex \bar{o} fixed by H . Let T_H be an H -subtree of T containing \bar{o} and every edge with H -finite stabiliser. The action of H on T_H is a trivial action, since it has a vertex fixed by H , and so the orbit space $H \backslash T_H$ is a tree, which might well be finite, but must have at least one edge. Our strategy is to show that if G is finitely generated over H and there is an H -almost invariant set A satisfying $AH = A$, then we can find a G -tree T with the required properties by first deciding what $H \backslash T_H$ must be and then lifting to get T_H and then T .

We show that if G is finitely generated over H , then there is a G -graph X if which there is a vertex with stabiliser H and in which a proper H -almost invariant set A satisfying $AH = A$ corresponds to a proper set of vertices with H -finite coboundary. It then follows from the theory of [5], described in the previous section, that there is a sequence of structure trees for $H \backslash X$. We choose one of these to be $H \backslash T_H$, and show that we can lift this to obtain T_H and then T itself.

For example if $G = H *_K L$ then there is a G -tree Y with one orbit of edges and a vertex \bar{o} fixed by H , and every edge incident with \bar{o} has H -finite stabiliser. Suppose that K, L are such that these are the only edges with H -finite stabilisers. Then $H \backslash T_H$ has two vertices and one edge. When we lift to T_H we obtain an H -tree of diameter two in which the middle vertex \bar{o} has stabiliser H . The tree T is covered by the translates of T_H .

On the other hand, if $G = L *_K H$ where K is finite, and T is as above, then every edge of T is H -finite and so T_H is T regarded as an H -tree. The fact that our construction gives a canonical construction for $H \backslash T_H$ means that when we lift to T_H and T we will get the unique tree that admits the action of G .

We proceed with our proof.

Lemma 3.1. *The group G is finitely generated over H if and only if there is a connected G -graph X with one orbit of vertices, and finitely many orbits of edges, and there is a vertex o with stabiliser H .*

Proof. Suppose G is generated by $H \cup S$, where S is finite. Let X be the graph with $VX = \{gH | g \in G\}$ and in which EX is the set of unordered pairs $\{\{gH, gsH\}, g \in G, s \in S\}$. We then have that X is vertex transitive, there is a vertex $o = H$ with stabilizer H and $G \backslash X$ is finite. We have to show that X is connected. Let C be the component of X containing o . Let G' be the set of those $g \in G$ for which $gH \in C$. Clearly $G'H = G'$ and $G's = G'$ for every $s \in S$. Hence $G' = G$ and $C = X$. Thus X is connected.

Conversely let X be a connected G -graph and $VX = Go$ where $G_o = H$. Suppose EX has finitely many G -orbits, Ge_1, Ge_2, \dots, Ge_r where e_i has vertices o and $g_i o$. It is not hard to show that G is generated by $H \cup \{g_1, g_2, \dots, g_r\}$. □

Let $A \subset G$ be a proper H -almost invariant set satisfying $AH = A$. Let G be finitely generated over H , and let X be a G -graph as in the last lemma. There is a subset of VX corresponding to A , which is also denoted A . For any $x \in G$, $A + Ax$ is H -finite. In particular this is true if $s \in S$. This means that δA is H -finite. Note that neither A nor $A^* = VX - A$ is H -finite. Thus a proper H -almost invariant set corresponds to a proper subset of VX such that δA is H -finite.

From the previous section (Lemma 2.2) we know that $\mathcal{B}(H \backslash X)$ has a uniquely determined nested set of generators $\mathcal{E} = \mathcal{E}(H \backslash X)$. For $E \in \mathcal{E}$, let $\bar{E} \subset VX$ be the set of all $v \in VX$ such that $Hv \in E$. Let C be a component of \bar{E} .

Lemma 3.2. *For $h \in H$, $hC = C$ or $hC \cap C = \emptyset$. Also $HC = \bar{E}$, $h\delta C \cap \delta C = \delta C$ or $h\delta C \cap \delta C = \emptyset$ and $H \backslash \delta C = \delta E$.*

Proof. Let $h \in H$. Then hC is also a component of \bar{E} , since $HC \subseteq E$. Thus either $hC = C$ or $hC \cap C = \emptyset$. Let K be the stabilizer of C in H . If $v \in C$ then $hv \in C$ if and only if $h \in K$. Thus $K \backslash C$ injects into $H \backslash C = E$ and $K \backslash \delta C$ injects into δE . But E is connected, and so the image HC is E . It follows that there is a single H -orbit of components. □

It follows from the lemma that it is also the case that C^* is connected, since any component of C^* must have coboundary that includes an edge from each orbit of δC . Let $\bar{\mathcal{E}}(H, X)$ be the set of all such C , and let $\bar{\mathcal{E}}_n(H, X)$ be the subset of $\bar{\mathcal{E}}(H, X)$ corresponding to those C for which δC lies in at most n H -orbits.

Lemma 3.3. *the set $\bar{\mathcal{E}}(H, X)$ is a nested set. The set $\bar{\mathcal{E}}_n(H, X)$ is the edge set of an H -tree.*

Proof. Let $C, D \in \bar{\mathcal{E}}_n(H, X)$. Then HC, HD are in the nested set \mathcal{E} . Suppose $HC \subset HD$, then $C \subset D$ or $C \cap D = \emptyset$. It follows easily that $\bar{\mathcal{E}}(H, X)$ is nested. It was shown in [4] that a nested set \mathcal{E} is the directed edge set of a tree if and only if it satisfies the finite interval condition, i.e. if $C, D \in \mathcal{E}$ and $C \subset D$, then there are only finitely many $E \in \mathcal{E}$ such that $C \subset E \subset D$. Thus we have to show that $\bar{\mathcal{E}}_n(H, X)$ satisfies the finite interval condition. If $C \subset D$ and $C \subseteq E \subseteq D$ where $C, E, D \in \bar{\mathcal{E}}_n(H, X)$, then $HC \subseteq HE \subseteq HD$. But $\mathcal{E}_n(H, X)$ does satisfy the finite interval condition and $HC = HE$ implies $C = E$. Now let $C \cap D = \emptyset$ and suppose that $o = H \in C^* \cap D^*$. There are only finitely many $E \in \bar{\mathcal{E}}_n$ such that $C \subset E$ and $o \in E^*$ or such that $D \subset E^*$ and $o \in E$. Each $E \in \bar{\mathcal{E}}_n$ such that $C \subset E \subset D^*$ has one of these two properties. □

Let $\bar{T} = \bar{T}(H)$ be the tree constructed in the last Lemma. Let $T = H \backslash \bar{T}$. Note that in the above $\bar{T}(H)$ is the Bass-Serre H -tree associated with the quotient graph $T(H) = H \backslash \bar{T}(H)$ and the graph of groups obtained by associating appropriate labels to the edges and vertices of this quotient graph (which is a tree). Clearly the action of H on $T(H)$ is a trivial action in that H fixes the vertex $\bar{o} = \nu o$. The stabilisers of edges or vertices on a path or ray beginning at \bar{o} will form a non-increasing sequence of subgroups of H .

We now adapt the argument of the previous section to show that if $A \subset VX$ is such that δA lies in at most n H -orbits, then there is a canonical way of expressing A in terms of the set $\mathcal{E}(H, X)$. In this case we have to allow unions of infinitely many elements of the generating set. Our induction hypothesis is that if δA lies in at most n H -orbits, then A is canonically expressed in terms of $\mathcal{E}_n(H, X)$. First note that there are only finitely many H -orbits of elements of $\mathcal{E}_n = \mathcal{E}_n(H, X)$ with which A is not nested. This is because if $C \in \mathcal{E}_n$ is not nested with A and F is a finite connected subgraph of $H \setminus X$ containing all the edges of $H\delta A$, then $H\delta C$ must contain an edge of F and there are only finitely many elements of \mathcal{E}_n with this property. We now let $\mu(A)$ be the number of H -orbits of elements of \mathcal{E}_n with which A is not nested. If $\mu(A) = 0$, then A is nested with every $C \in \mathcal{E}_n$. This then means that if neither A nor A^* is empty and it is not already in \mathcal{E}_n , then A determines a vertex z of \bar{T}_n and either A or A^* is the union (possibly infinite) of edges of T_n that lie in finitely many H -orbits. If A is such a union, then we use this union for our canonical expression for A . If A is not such a union, then A^* is; we have $A^* = \bigcup \{C_\lambda | \lambda \in \Lambda\}$, where each C_λ has $\tau C_\lambda = z$ and the edges lie in finitely many H -orbits. We write $A = (\bigcup \{C_\lambda | \lambda \in \Lambda\})^* = \bigcap \{C_\lambda^* | \lambda \in \Lambda\}$. Note that this gives a canonical way of expressing cuts corresponding to a vertex that is not in the image of ν and whose incident edges lie in finitely many H -orbits. Suppose then that the hypothesis is true for elements B for which $\mu(B) < \mu(A)$. Let $C \in \mathcal{E}_n$ be not nested with A . Then $\mu(A \cap HC) + \mu(A \cap HC^*) \leq \mu(A)$. Thus each of $A \cap HC$ and $A \cap HC^*$ can be expressed in a unique way in terms of the \mathcal{E}_n . We take the expression for A to be the union of the two expressions for $A \cap HC$ and $A \cap HC^*$ except that we include hC for $h \in H$, only if just one of the two expressions involve hC .

If $g \in G$, then $g\bar{T}(H)$ is a (gHg^{-1}) -tree. It is the tree $\bar{T}(gHg^{-1})$ obtained from the G -graph X by using the vertex go instead of o . We now show that there is a G -tree T which contains all of the trees $g\bar{T}(H)$.

We know that the action of the group G on X is vertex transitive and that X has a vertex o fixed by H . Also G is generated by $H \cup S$ where S is finite.

Clearly there is an isomorphism $\alpha_g : \bar{T}(H) \rightarrow \bar{T}(gHg^{-1})$ in which $D \mapsto gD$.

Suppose now that $\nu o \neq \nu(go)$. Let A, B be H -almost invariant sets satisfying $AH = A, BH = B$ and let $g \in G$. We regard A, B as subsets of VX , so that δA and δB are H -finite.

Suppose that $o \in gB^*$ and $go \in A^*$. The following Lemma is due to Kropholler [8], [9]. We put $K = gHg^{-1}$.

Lemma 3.4. *In this situation $\delta(A \cap gB)$ is $(H \cap K)$ -finite.*

Proof. Let $x \in G$. We show that the symmetric difference $(A \cap gB)x + (A \cap gB)$ is $(H \cap K)$ -finite. Since A, B are H -almost invariant, there are finite sets E, F such that $A + Ax \subseteq HE$ and $B + Bx \subseteq HF$. We then have

$$(A \cap gB)x + (A \cap gB) = Ax \cap (gBx + gB) + (Ax + A) \cap gB = Ax \cap gHF + g(g^{-1}HE \cap B).$$

Now $Ax \cap gHF$ is K -finite, but it is also H -finite because gH is contained in A^* , since $go \in A^*$. A set which is both H -finite and K -finite is $H \cap K$ -finite. Thus $Ax \cap gHF$ is $(H \cap K)$ -finite. Similarly using the fact that $g^{-1}o \in B^*$, it follows that $g^{-1}HE \cap B$ is $H \cap (g^{-1}Hg)$ -finite, and so $g(g^{-1}HE \cap B)$ is $(H \cap K)$ -finite. Thus $A \cap gB$ is $(H \cap K)$ -almost invariant. But this means that $\delta(A \cap gB)$ is $(H \cap K)$ -finite. \square

What this Lemma says is that if A, gB are not nested then there is a special corner - sometimes called the *Kropholler corner* - which is $(H \cap K)$ -almost invariant.

Notice that in the above situation all of $\delta A, \delta(A \cap gB^*)$ and $\delta(A \cap gB)$ are H -finite. If we take the canonical decomposition for A , then it can be obtained from the canonical decompositions for $A \cap gB$ and $A \cap gB^*$ by taking their union and deleting any edge that lies in both. Also $\delta(gB)$ is K -finite and the decomposition for gB can be obtained from those for $gB \cap A$ and $gB \cap A^*$. But the edges in the decomposition for $A \cap gB$ which is $(H \cap K)$ -almost invariant are the same in both decompositions.

We will now show that it follows from Lemma 3.4 that the set $G\bar{\mathcal{E}}_n$ is a nested G -set which satisfies the final interval condition, and so it is the edge set of a G -tree. We have seen that $\bar{\mathcal{E}}_n$ is a nested H -set where $\mathcal{E}_n = H \setminus \bar{\mathcal{E}}_n$ is the uniquely determined nested subset of $\mathcal{B}_n(H \setminus X)$ that generates $\mathcal{B}_n(H \setminus X)$ as an abelian group. It is the edge set of a tree $T_n(H \setminus X)$.

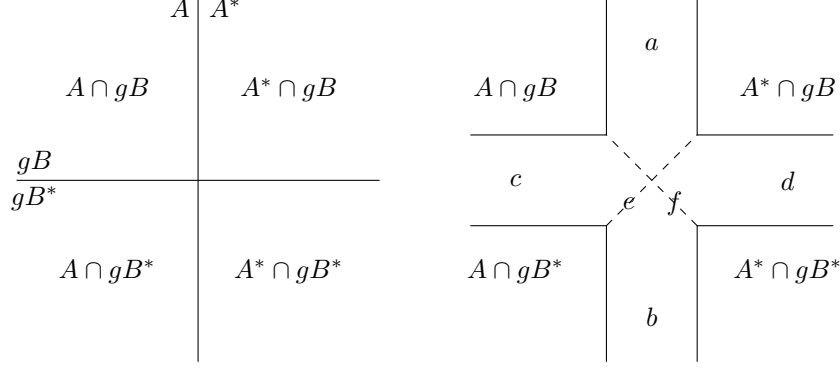


FIGURE 1. Crossing cuts

If $A, B \in \bar{\mathcal{E}}_n$ and A, gB are not nested for some $g \in G$, then by Lemma 3.4 there is a corner -the Kropholler corner -, which we take to be $A \cap gB$, for which $\delta(A \cap gB)$ is $(H \cap K)$ -finite. We then have canonical decompositions for $A \cap gB$ and $A \cap gB^*$ as above. This is illustrated in Fig 1. The labels a, b, c, d, e, f are for sets of edges joining the indicated corners. In this case the letters do not represent edges of X but elements of $\bar{\mathcal{E}}_n$. Although each $E \in \bar{\mathcal{E}}_n$ comes with a natural direction, in the diagram we only count the unoriented edges, i.e. we count the number of edge pairs (E, E^*) . In the diagram, $A \cap gB$ is always taken to be the Kropholler corner. Thus we have that any pair contributing to a, f or e must be $(H \cap K)$ -finite. Any pair contributing to e or b must be H -finite and any pair contributing to e or d must be K -finite.

We have that $a + e + f + b = 1$ and $c + e + f + d = 1$. Suppose that the Kropholler corner $A \cap B$ is not empty. It is the case that each of o and go lies in one of the other three corners. We know that $o \in gB^*, go \in A^*$. If $o \in A \cap gB^*$ and $go \in A^* \cap gB$, then $a = c = 1$ and $e = f = b = d = 0$ and $A^* \cap gB^* = \emptyset$. If $o \in A^* \cap gB$ and $go \in A^* \cap gB^*$, then $a = d = 1$ and $A \cap gB^* = \emptyset$, while if both o and go are in $A^* \cap gB^*$, then either $a = d = 1$ and $A \cap gB^* = \emptyset$ or $a = c = 1$ and $A^* \cap gB = \emptyset$ or $f = 1$ and both $A \cap gB^*$ and $A^* \cap gB$ are empty, so that $A = gB$. In all cases A, gB are nested.



We need also to show that $G\bar{\mathcal{E}}_n$ satisfies the finite interval condition. Let $g \in G$ and let $K = gHg^{-1}$. Consider the union $\bar{\mathcal{E}} \cup g\bar{\mathcal{E}}$. This will be a nested set. In fact it will be the edge set of a tree that is the union of the trees $T(H)$ and $T(K)$. In the diagram the red edges are the edges that are just in $T(H)$. The blue edges are the ones that are in $T(K)$. The brown edges are in both $T(H)$ and $T(K)$. An edge is in the geodesic joining o and go if and only if it has stabiliser containing $H \cap K$, it will also lie in both $T(H)$ and $T(K)$ (i.e. it is coloured brown) if and only if its stabiliser contains

$H \cap K$ as a subgroup of finite index. It may be the case that $T(H)$ and $T(K)$ have no edges in common, i.e. there are no brown edges. An edge lies in both trees if and only if it has a stabiliser that is $(H \cap K)$ -finite. If there are such edges then they will be the edge set of a subtree of both trees. They will correspond to the edge set $\bar{\mathcal{E}}(H \cap K)$.

It follows that $T(H)$ is always a subtree of a tree constructed from a subset of $G\bar{\mathcal{E}}_n$ that contains $\bar{\mathcal{E}}_n$. If $T(H)$ and $T(K)$ do have an edge in common, then $T(H) \cup T(K)$ will be a subtree of the tree we are constructing. If $e \in EX$ has vertices go and ko and there is some $C \in G\bar{\mathcal{E}}_n$ that has $e \in \delta C$, then $C \in gET(g^{-1}Hg) \cap kET(k^{-1}Hk)$. If there is no such C , i.e. there is no cut $C \in G\bar{\mathcal{E}}_n$ that separates o and $k^{-1}go$ then $T(H) = k^{-1}gT(H)$. As there is a finite path connecting any two vertices u, v in X , it can be seen that there are only finitely many edges in $G\bar{\mathcal{E}}_n$ separating u and v since any such edge must separate the vertices of one of the edges in the path. Thus $G\bar{\mathcal{E}}_n$ is the edge set of a tree.

We say that a G -tree T is reduced if for every $e \in ET$, with vertices ιe and τe we have that either ιe and τe are in the same orbit, or G_e is a proper subgroup of both $G_{\iota e}$ and $G_{\tau e}$.

Theorem 3.5. *Let G be a group that is finitely generated over a subgroup H . The following are equivalent:-*

- (i) *There is a proper H -almost invariant set $A = HAK$ with left stabiliser H and right stabiliser K , such that A and gA are nested for every $g \in G$.*
- (ii) *There is a reduced G -tree T with vertex v and incident edge e such that $G_v = K$ and $G_e = H$.*

Proof. It is shown that (ii) implies (i) in the Introduction.

Suppose then that we have (i). We will show that there is a G -tree - in which G acts on the right - which contains the set $V = \{Ax | x \in G\}$ as a subset. Let $x \in G$, then $A + Ax$ is a union of finitely many coset $\{Hg_1, Hg_2, \dots, Hg_k\}$. Then $\{g_1^{-1}A, g_2^{-1}A, \dots, g_k^{-1}A\}$ is the edge set of a finite tree F . We know that the set $\{gA | g \in G\}$ is the edge set of a G -tree T provided we can show that it satisfies the finite interval condition. But this must be the case as the edges separating vertices A and Ax will be the edges of F . □

Theorem 3.6. *Let G be a group and let H be a subgroup, and suppose G is finitely generated over H . There is a proper H -almost invariant subset A such that $A = AH$, if and only if there is a non-trivial reduced G -tree T in which H fixes a vertex and every edge orbit contains an edge with an H -finite edge stabilizer.*

Proof. The only if part of the theorem is proved in Theorem 3.5. In fact it is shown there that if G has an action on a tree with the specified properties, then there is a proper H almost invariant set A for which $HAH = A$.

Suppose then that G has an H -almost invariant set A such that $AH = A$. Since G is finitely generated over H , we can construct the G -graph X as above, in which A can be regarded as a set of vertices for which δA lies in finitely many H -orbits. Let this number of orbits be n . Then we have seen that there is a G -tree \bar{T}_n for which H fixes a vertex \bar{o} and every edge is in the same G -orbit as an edge in $\bar{T}(H)$. The edges in this tree are H -finite. The set A has an expression in terms of the edges of $\bar{T}(H)$. Finally we need to show that the action on \bar{T}_n is non-trivial. If G fixes \bar{o} , then $\nu(A)$ consists of the single vertex o and so A is not proper. In fact the fact that A is proper ensures that no vertex of \bar{T}_n is fixed by G .

It can be seen from the above that $\bar{T}(H) \cap \bar{T}(g^{-1}Hg) = \bar{T}(H \cap gHg^{-1})$ so that if $e \in ET(H)$, and $g \in G_e$, then $e \in \bar{T}(gHg^{-1})$ and so G_e is H -finite. □

The Kropholler Conjecture follows immediately from the last Theorem.

4. H -ALMOST STABILITY

Let G be a group with subgroup H , and let T be a G -tree.

Let $\bar{A} \subset VT$ be such that $\delta\bar{A} \subset ET$ consists of finitely many H -orbits of edges e such that G_e is H -finite. Also let H fix a vertex of T . Note that $\delta\bar{A}$ consists of whole H -orbits, so that $e \in \delta\bar{A}$

implies $he \in \delta\bar{A}$ for every $h \in H$. The fact that G_e is H -finite for $e \in \delta\bar{A}$ follows from the fact that $\delta\bar{A}$ is H -finite. If H_e is the stabiliser of $e \in \delta\bar{A}$, then $[G_e : H_e]$ is finite.

Let $v \in VT$, and let $A = A(v) = \{g \in G | gv \in \bar{A}\}$. Note that $A(xv) = A(v)x^{-1}$, so that the left action on T becomes a right action on the sets $A(v)$. If $x \in G$ and $[v, xv]$ is the geodesic from v to xv , then $g \in A + Ax$ if and only if the geodesic $[gv, gxv]$ contains an odd number of edges in $\delta\bar{A}$. If $[v, xv]$ consists of the edges e_1, e_2, \dots, e_r , then $ge_i \in \delta\bar{A}$ if and only if $Hge_i \in \delta\bar{A}$. It follows that $H(A + Ax) = A + Ax$. It is also clear that for each e_i there are only finitely many cosets Hg such that $Hge_i \in \delta\bar{A}$. Thus A is H -almost invariant. We also have $A(v)H = A(v)$ if H fixes v .

For each $e \in ET$, let $d(e)$ be the number of cosets Hg such that $Hge \in \delta\bar{A}$. We see that $d(e) = d(xe)$ for every $x \in G$ and so we have a metric on VT , that is invariant under the action of G . We will show that if G has an H -almost invariant set such that $HAH = A$ then there is a G -tree with a metric corresponding to this set.

From now on we are interested in the action of G on the set of H -almost invariant sets. But note that we are interested in the action by right multiplication. The Almost Stability Theorem [3], also used the action by right multiplication. Let $A \subset G$ be H -almost invariant and let $HA = A$. For the moment we do not assume that $AH = A$.

Let $M = \{B | B =_a A\}$ so that for $B, C \in M$, $B + C = HF$ where F is finite.

Note that for $H = \{1\}$ it follows from the Almost Stability Theorem that M is the vertex set of a G -tree.

We define a metric on M . For $B, C \in M$ define $d(B, C)$ to be the number of H -cosets in $B + C$.

This is a metric on M , since $(B + C) + (C + D) = (B + D)$, and so an element which is in $B + D$ is in just one of $B + C$ or $C + D$. Thus $d(B, D) \leq d(B, C) + d(C, D)$.

Also G acts on M by right multiplication and this action is by isometries, since $(B + C)z = Bz + Cz$. Let Γ be the graph with $VT = M$ and two vertices are joined by an edge if they are distance one apart. Every edge in Γ corresponds to a particular H -coset. There are exactly $n!$ geodesics joining B and C if $d(B, C) = n$, since a geodesic will correspond to a permutation of the cosets in $B + C$. The vertices of Γ on such a geodesic form the vertices of an n -cube.

The edges corresponding to a particular coset Hb disconnect Γ , since removing this set of edges gives two sets of vertices, B and B^* , where B is the set of those $C \in M$ such that $Hb \subset C$.

It has been pointed out to me by Graham Niblo that Γ is the 1-skeleton of the Sageev cubing introduced in [12]. For completeness we describe this alternative characterization of Γ .

Let G be a group with subgroup H and let $A = HA$ be an H -almost invariant subset. Let

$$\Sigma = \{gA | g \in G\} \cup \{gA^* | g \in G\}.$$

We define a graph Γ' . A vertex V of Γ' is a subset of Σ satisfying the following conditions:-

- (1) For all $B \in \Gamma'$, exactly one of B, B^* is in V .
- (2) If $B \in V, C \in \Sigma$ and $B \subseteq C$, then $C \in V$.

Two vertices are joined by an edge in Γ' if they differ by one element of Σ . For $g \in G$, there is a vertex V_g consisting of all the elements of Σ that contain g . Then Sageev shows that there is a component Γ^1 of Γ' that contains all the V_g . In fact this graph Γ^1 is isomorphic to our Γ .

By (1) for each $V \in \Sigma$ either $A \in V$ or $A^* \in V$ but not both. Let Σ_A be the subset of Σ consisting of those $V \in \Sigma$ for which $A \subset V$. The edges joining Σ_A and Σ_A^* in Γ^1 form a hyperplane. Each edge in the hyperplane joins a pair of vertices that differ only on the set A . For each xA there is a hyperplane joining vertices that differ only on the set xA . Clearly G acts transitively on the set of hyperplanes.

With V as above, consider the subset A_V of G

$$A_V = \{x \in G | x^{-1}A \in V\}.$$

Then $HA_V = A_V$ and $A_{V_1} = A$. Also $A_V + A$ is the union of those cosets Hx for which V and V_1 differ on $x^{-1}A$, which is finite. Thus $A_V \in VT$.

Thus there is a map $VT^1 \rightarrow VT$ in which $V \mapsto A_V$. This map is a G -map and an isomorphism of graphs.

If the set A is such that A and gA are nested for every $g \in G$, then there is a G -subgraph of Γ_1 which is a G -tree. This will also be true of Γ .

In Γ a hyperplane consists of edges joining those vertices that differ only by a particular coset Hx . Every edge of Γ belongs to just one hyperplane. The group G acts transitively on hyperplanes. The hyperplane corresponding to Hx has stabilizer $x^{-1}Hx$.

Suppose now that A is H -almost invariant with $HAK = A$. Here H is the left stabiliser and K is the right stabiliser of A , and we assume that $H \leq K$, so that in particular $HAH = A$. Note that it follows from the fact that A is H -almost invariant that it is also K almost invariant. Suppose that G is finitely generated over K . We have seen, in the previous section, that there is a G -tree T in which A uniquely determines a set \bar{A} of vertices with H -finite coboundary $\delta\bar{A}$. Here $T = T_n$ for n sufficiently large that in the graph X - as defined in the previous section - the set $\delta\bar{A}$ is contained in at most n H -orbits of edges. Note that if e is an edge of $\bar{T}(H) = \bar{\mathcal{E}}(H, X)$, then δe is H_e -finite, and will consist of finitely many H_e -orbits. It is then the case that $[G_e : H_e]$ is finite, since δe will consist of finitely many G_e -orbits each of which is a union of $[G_e : H_e]$ H_e -orbits of edges.

We also know that K fixes a vertex \bar{o} of T , and that $H\delta\bar{A} = \delta\bar{A}$. Thus $\delta\bar{A}$ consists of finitely many H -orbits of edges. We can contract any edge whose G -orbit does not intersect $\delta\bar{A}$. We will then have a tree that has the properties indicated at the beginning of this section. Thus $\bar{A} \subset VT$ is such that $\delta\bar{A} \subset ET$ consists of finitely many H -orbits of edges e such that G_e is H -finite. We see that the metric d on M is the same as the metric defined on VT . Explicitly we have proved the following theorem in the case when G is finitely generated over K .

Theorem 4.1. *Let G be a group with subgroup H and let $A = HAK$ where $H \leq K$ and A is H -almost invariant. Let M be the G -metric space defined above. Then there is a G -tree T such that VT is a G -subset of M and the metric on M restricts to a geodesic metric on VT . If $e \in ET$ then some edge in the G -orbit of e has H -finite stabiliser.*

This is illustrated in Fig 1 and Fig 2.

Proof. It remains to show that the theorem for arbitrary G follows from the case when G is finitely generated over K . Thus if F is a finite subset of G , then there is a finite convex subgraph C of Γ containing AF . We can use the graph X of the previous section for the subgroup L of G generated by $H \cup F$ to construct an L -tree which has a subtree $S(F)$ with vertex set contained in VC . These subtrees have the nice property that if $F_1 \subset F_2$ then $S(F_1)$ is a subtree of $S(F_2)$. They therefore fit together nicely to give the required G -tree. We give a more detailed argument for why this is the case. We follow the approach of [1].

Let M' be the subspace of M consisting of the single G -orbit AG . Define an inner product on M' by $(B.C)_A = \frac{1}{2}(d(A, B) + d(A, C) - d(B, C))$.

This turns M' into a 0-hyperbolic space, i.e. it satisfies the inequality

$$(B.C)_A \geq \min\{(B.D)_A, (C.D)_A\}$$

for every $B, C, D \in M'$. This is because we know that if $L \leq G$ is finitely generated over H , then there is an L -tree which is a subspace of M . But A, B, C, D are vertices of such a subtree which is 0-hyperbolic. It now follows from [1], Chapter 2, Theorem 4.4 that there is a unique \mathbb{Z} -tree VT (up to isometry) containing M' . The subset of VT consisting of vertices of degree larger than 2 will be the vertices of a G -tree and can be regarded as a G -subset of M containing M' . \square

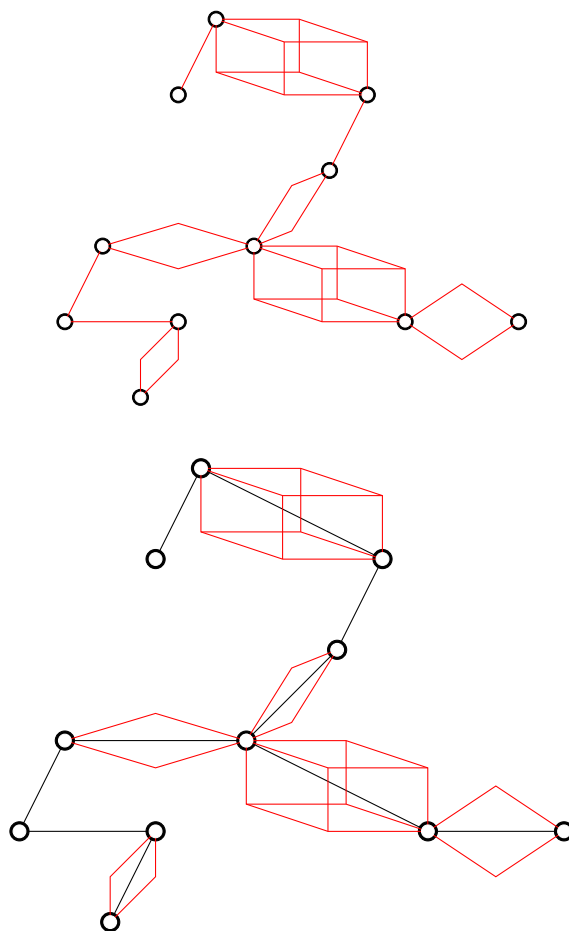


Fig 1

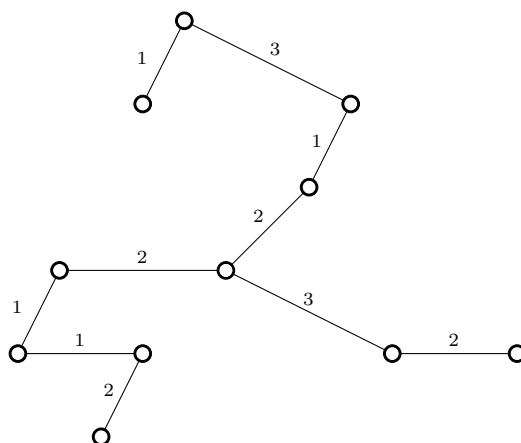


Fig 2

REFERENCES

- [1] Ian Chiswell. *Introduction to Λ -trees*. World Scientific, 2001.
- [2] Warren Dicks, *Group, trees and projective modules*, Springer Lecture Notes **790** 1980
- [3] Warren Dicks and M.J.Dunwoody, *Groups acting on graphs*, Cambridge University Press, 1989. Errata <http://mat.uab.es/~dicks/>
- [4] M.J.Dunwoody, *Accessibility and groups of cohomological dimension one*, Proc. London Math. Soc. **38** (1979) 193-215.
- [5] M.J. Dunwoody, *Structure trees and networks*, arXiv:1311.3929.
- [6] M.J.Dunwoody and M.Roller, *Splitting groups over polycyclic-by-finite subgroups*, Bull. London Math.Soc. **23** 29-36 (1989).
- [7] A.Kar and G.A.Niblo, *Relative ends ℓ^2 -invariants and property T*, arXiv:1003.2370.
- [8] P.H.Kropholler, *An analogue of the torus decomposition theorem for certain Poincaré groups*, Proc. London Math. Soc. (3) **60** 503-529 (1990).
- [9] P.H.Kropholler, *A group theoretic proof of the torus theorem*, London Math. Soc. Lecture Note Series **181** 138-158 (1991).
- [10] G.A. Niblo, *A geometric proof of Stallings theorem on groups with more than one end*, Geometriae Dedicata **105**, 61-76 (2004).
- [11] G. Niblo, M. Sageev, *The Kropholler conjecture*, In *Guido's Book of Conjectures*, Monographies de L'Enseignement Mathématique, 40. L'Enseignement Mathématique, Geneva, 2008.
- [12] M.Sageev, *Ends of group pairs and non-positively curved cube complexes*, Proc. London Math. Soc. (3) **71**, 585-617 (1995).
- [13] J.R. Stallings, *Group theory and three-dimensional manifolds*. Yale Mathematical Monographs,**4**. Yale University Press, New Haven, Conn.-London, 1971.